Mathematical Proofs

Announcements

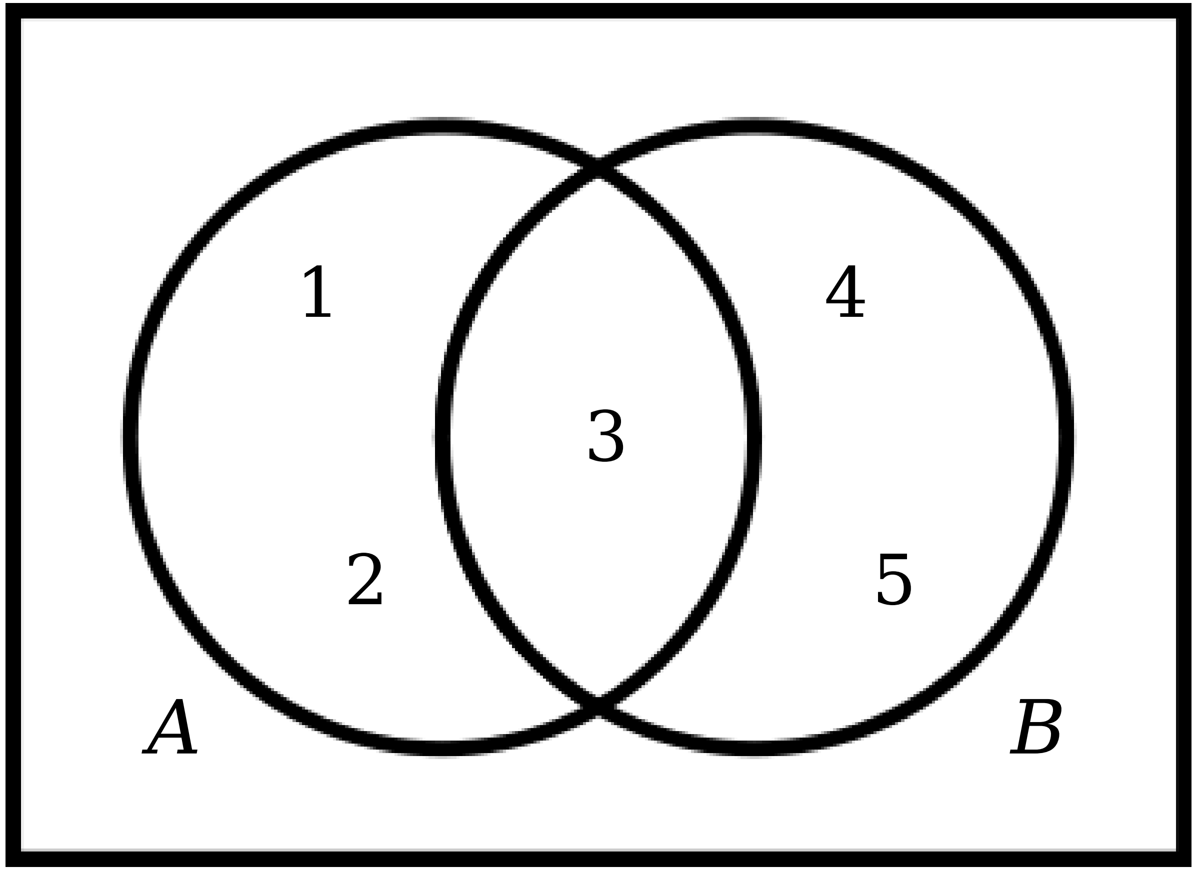
- ***Pset 0***
  - Due Friday
- ***Pset 1***
  - Goes out Friday, due following Friday
  - LaTeX Beginner's Quick Start Tutorial (LaTeX is the preferred tool for writing homework in this class)
- ***Office Hours***
  - They start Monday! Schedule will be on the course website by Friday. They will be accessible in person or by Zoom.

Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

Combining Sets

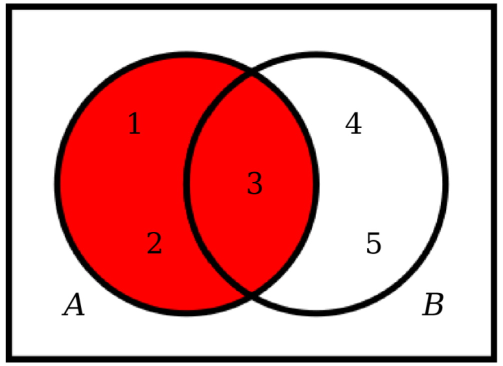
Venn Diagrams



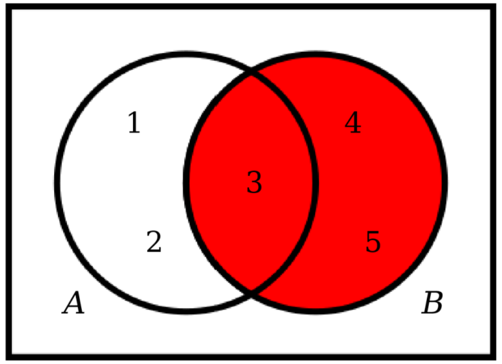
$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

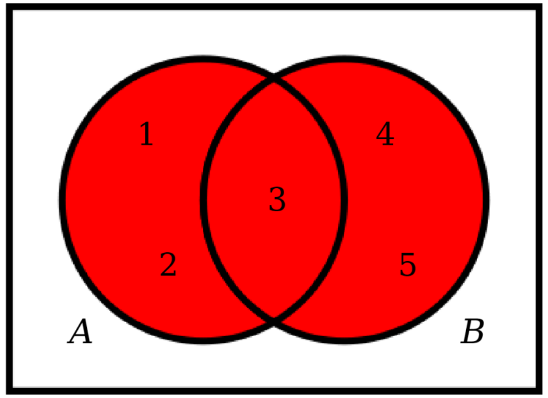
Venn Diagrams

A



B

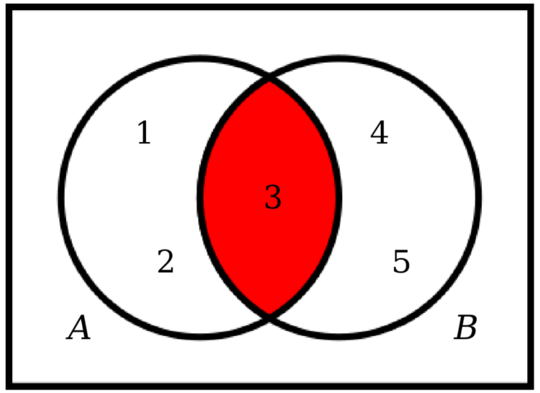




Union

$A \cup B$

{ 1, 2, 3, 4, 5 }



Intersection

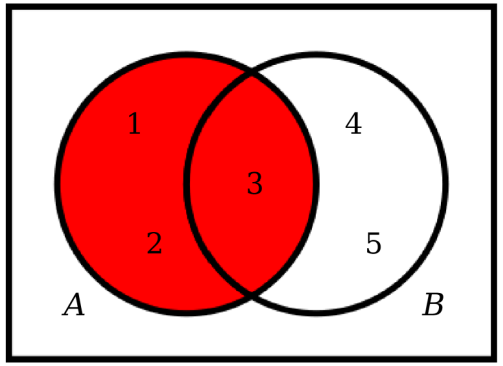
$A \cap B$

{ 3 }

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

Venn Diagrams

A



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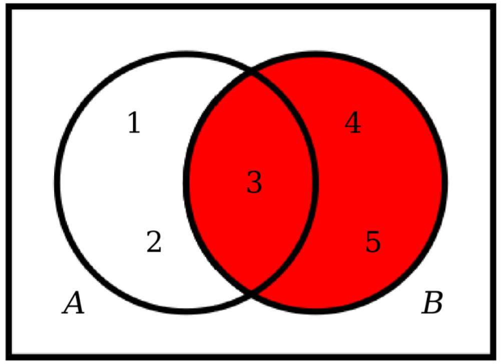
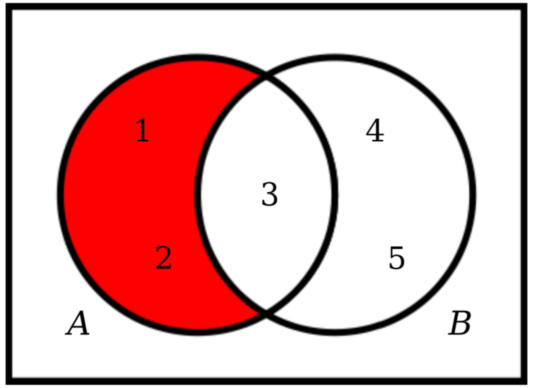


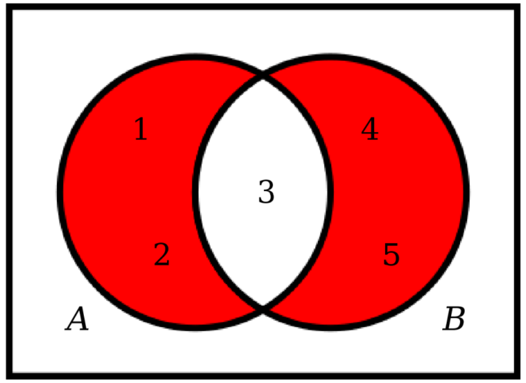


Difference

$A - B$

$A \setminus B$

$\{ 1, 2 \}$



Symmetric Difference

$A \Delta B$

$\{ 1, 2, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
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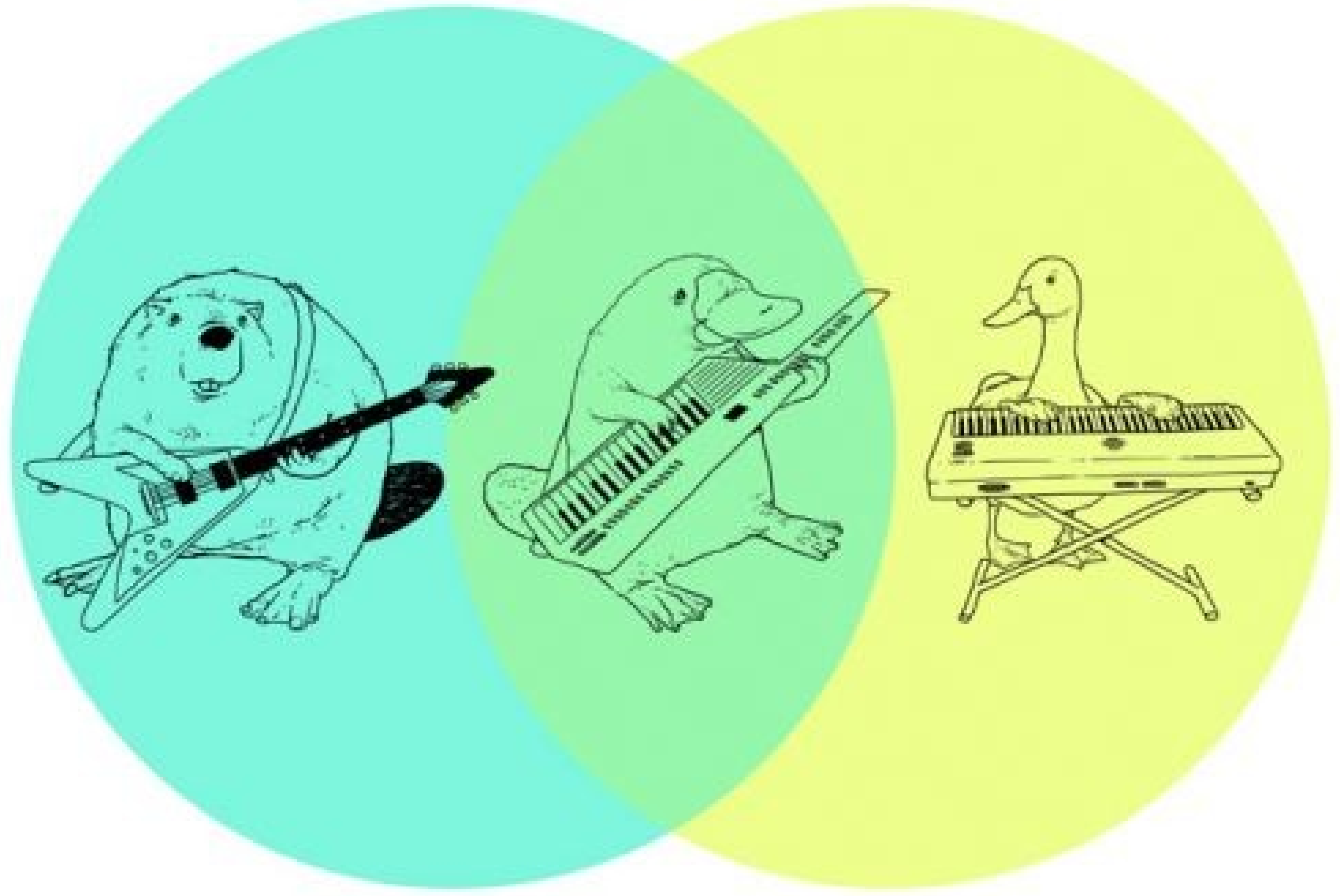
Venn Diagrams



What is a Proof?

A ***proof*** is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.

Proofwriting is not like other forms of writing, or even other forms of math problems, and we understand this can be a big adjustment! Here is some advice from years of teaching proofwriting.

Rule: Proofs are meant to argue something precisely and completely.

Advice: Well trained readers should find this persuasive, however *precision* and *completeness* are the goals you should have in mind, not *persuasion* in the usual social-emotional-rhetorical way that we think of persuasion.

Rule: Skipping even one step of a proof is a big deal—it makes the proof logically invalid.

Advice: Think of a proof as written driving directions from Point A to Point B, for someone who has never been, and who doesn't have a GPS/phone. If you leave out one left turn onto Main St., the driver will never get to the next right turn onto Maple St. They simply can't continue, and will be *permanently lost*! Having "some of the right intuition" isn't enough.

Writing our First Proof

Definitions:

An integer n is called ***even*** if
there is an integer k where $n = 2k$.

An integer n is called ***odd*** if
there is an integer k where $n = 2k + 1$.

Theorem: For all integers n, if n is even, then n^2 is even.

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“For all” means we need this to be true for all examples of elements of the set integers. So we challenge our reader: “pick ANY integer!!”

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The “if...then...” means that the “then” part only holds when the “if” part is true. So we are going to tell our reader to pick any integer **assuming** that the “if” condition is true of it. In other words, only pick even integers.

***Theorem:* For all** integers n, **if** n is even, **then** n^2 is even.

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Our First Proof!

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Let's Try Some Examples!

$$2^2 = 4 = 2 \cdot \mathbf{2}$$

$$10^2 = 100 = 2 \cdot \mathbf{50}$$

$$0^2 = 0 = 2 \cdot \mathbf{0}$$

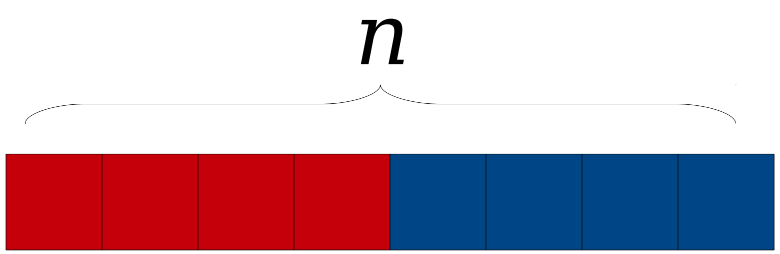
$$(-8)^2 = 64 = 2 \cdot \mathbf{32}$$

$$n^2 = 2 \cdot \mathbf{?}$$

What's the pattern? How do we predict this?

Theorem: For all integers n, if n is even, then n2 is even.

Let’s Draw Some Pictures!



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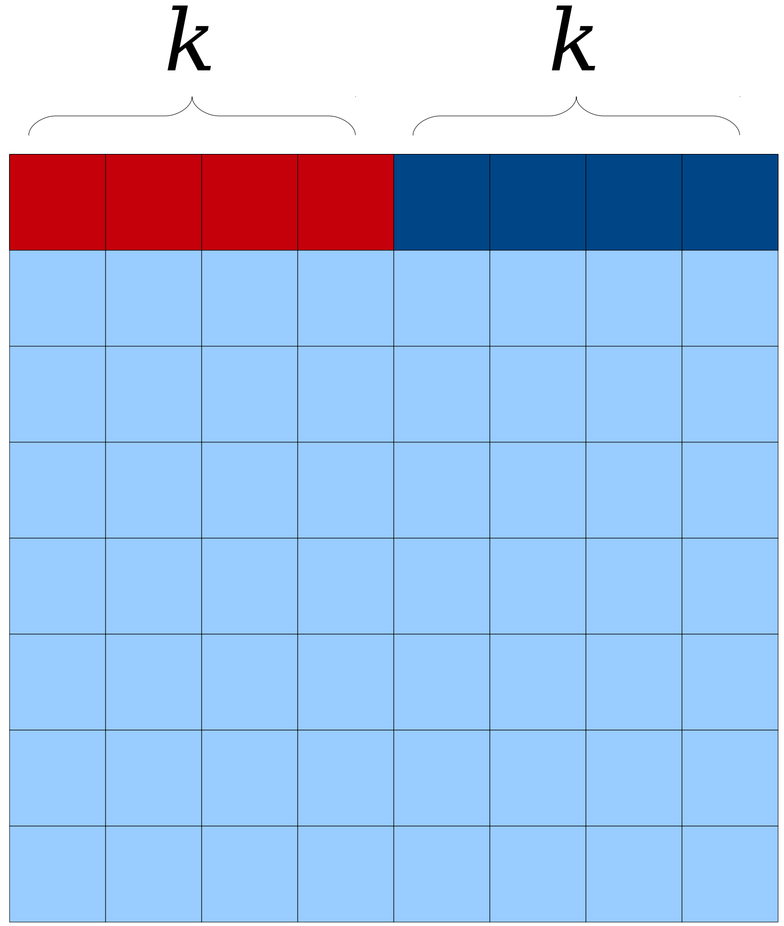

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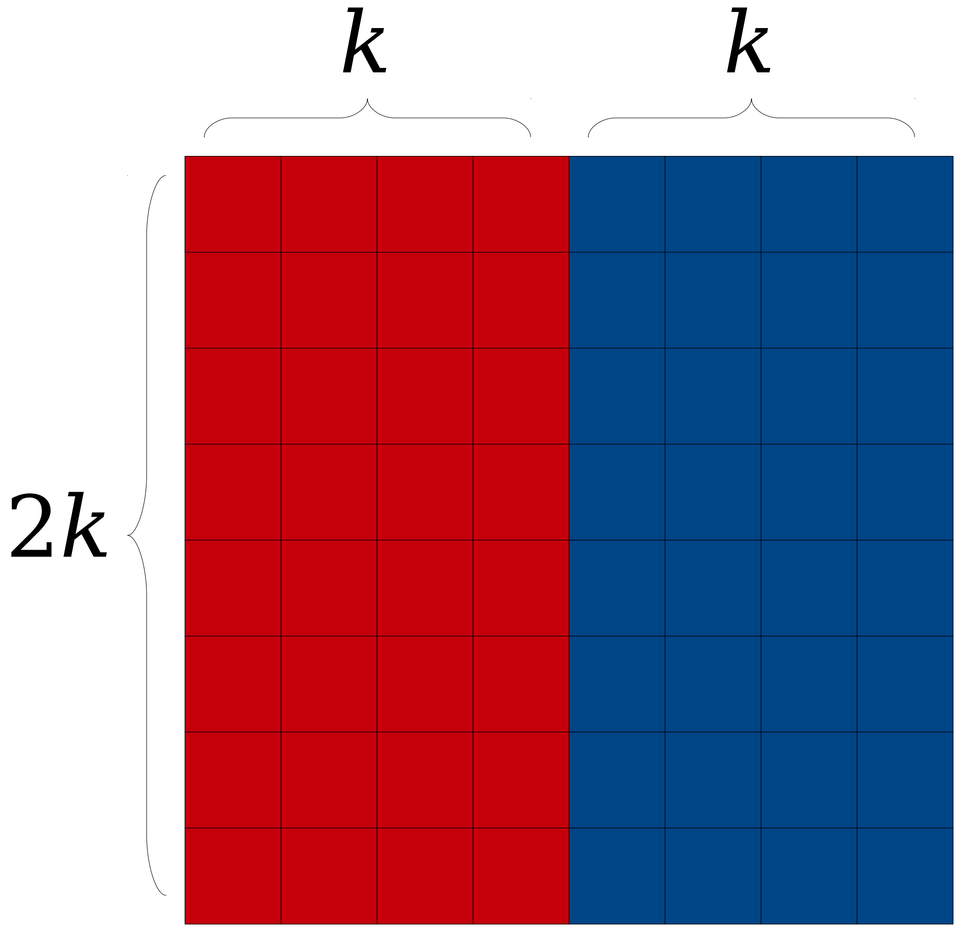

$n^2 = 2(2k^2)$

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From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$.

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This symbol means "end of proof." It's basically math nerd "mic drop."

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After assuming ***P*** is true, you need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

This is basically starting your driving directions by announcing what "Point A" and "Point B" are. Then you list the turn-by-turn directions that connect the two.

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Our Next Proof

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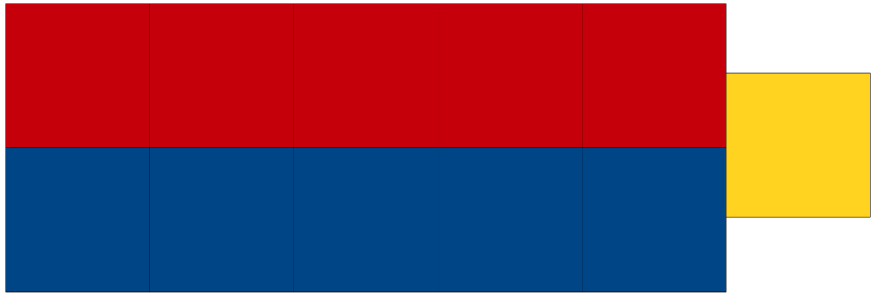
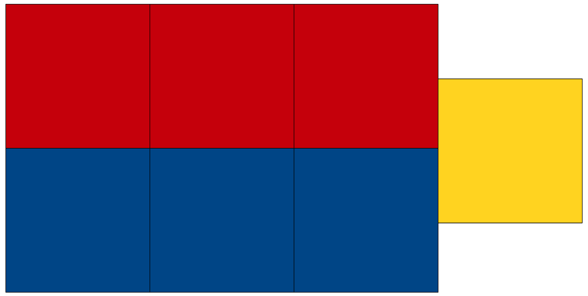
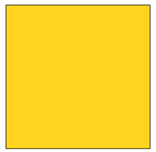
11  $2 \cdot \mathbf{5} + \mathbf{1}$

7  $2 \cdot \mathbf{3} + \mathbf{1}$

1  $2 \cdot \mathbf{0} + \mathbf{1}$

An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For all integers m and n,
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Question: How many of these are good first sentences of our proof?

- ***We will show that if m and n are odd, then m + n is even.***
- ***Pick arbitrary integers n and m.***
- ***Consider n=7 and m=3.***
- ***Let n and m be arbitrary integers.***
- ***Since n and m are odd, there are integers k and r such that n=2k + 1 and m = 2r + 1.***

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$$m = 2k + 1. \tag{1}$$

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Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \quad (1)$$

Similarly, because n is odd there must be some integer r such that

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that

By adding equa

We ask the reader to make an ***arbitrary choice***. Rather than specifying what m and n are, we're signaling to the reader that they could, in principle, supply any choices of m and n that they'd like.

By letting the reader pick m and n arbitrarily, anything we prove about m and n will generalize to all possible choices for those values.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

$$(2)$$

…earn that

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Theorem: For all integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We want to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

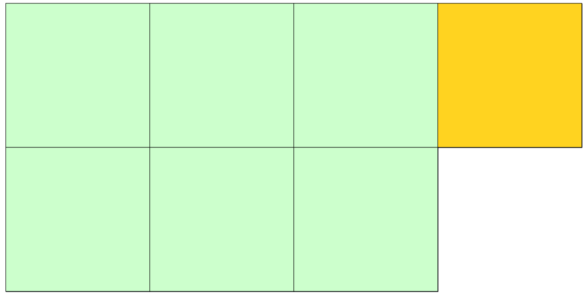
$$n = 2r + 1. \tag{2}$$

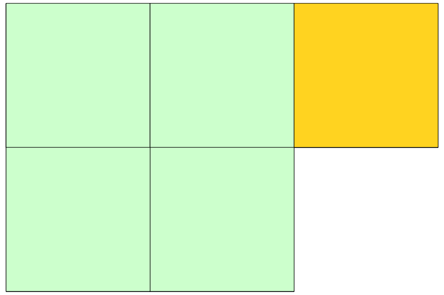
By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \tag{3}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Let's Draw Some Pictures!





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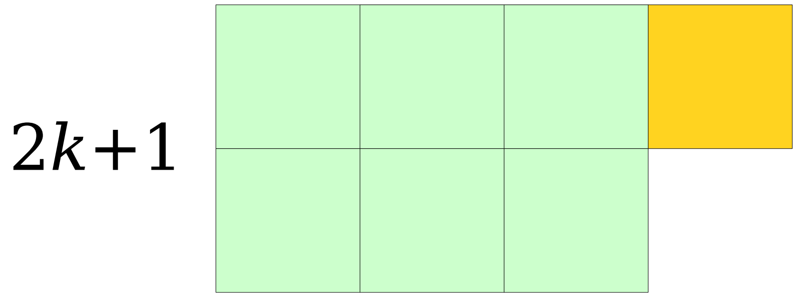
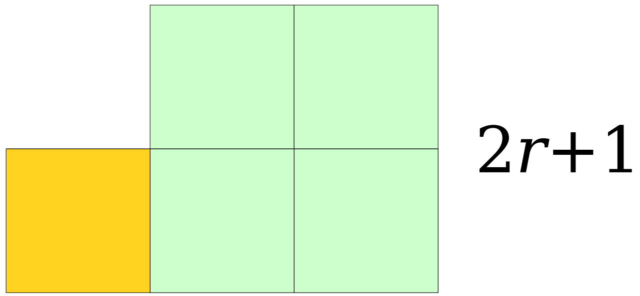
Let's Draw Some Pictures!

Theorem: For all integers m and n, if m and n are odd, then $m+n$ is even.

Let's Do Some Math!





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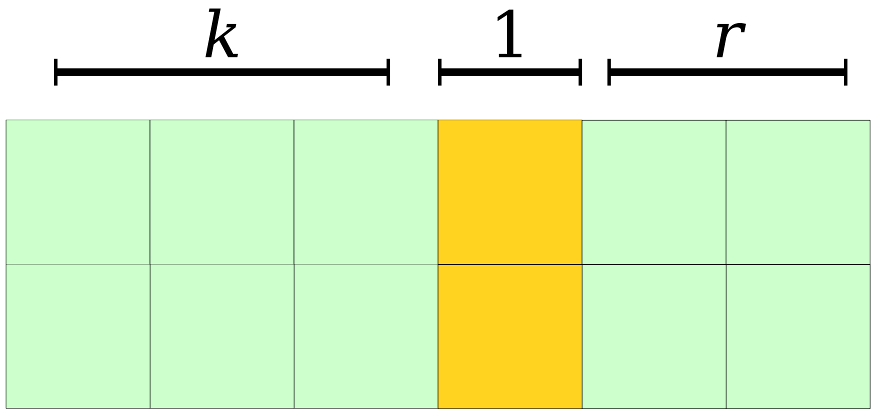
Let's Do Some Math!



$2k+1$ $2r+1$

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

Theorem: For all integers m and n, if m and n are odd, then $m+n$ is even.

Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For all odd integers n,
there exist integers r and s where $r^2 - s^2 = n$.

Theorem: For all odd integers n,
there exist integers r and s where $r^2 - s^2 = n$.

This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For all odd integers n,
there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that ***somewhere out there*** are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Approach:*** Search far and wide, find a *concrete example value* for *x* that has the right property. In the proof, (1) announce the find to your reader, *then* (2) show why your choice is correct.

Let's Try Some Examples!

$1 = \underline{\quad}^2 - \underline{\quad}^2$

$3 = \underline{\quad}^2 - \underline{\quad}^2$

$5 = \underline{\quad}^2 - \underline{\quad}^2$

$7 = \underline{\quad}^2 - \underline{\quad}^2$

$9 = \underline{\quad}^2 - \underline{\quad}^2$

Theorem: For all odd integers *n*,
there exist integers *r* and *s* where $r^2 - s^2 = n$.

Let's Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

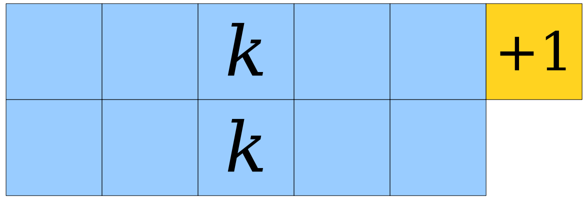
$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

We've got a pattern – but why does this work?

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Draw Some Pictures!



Theorem: For all odd integers n,
there exist integers r and s where $r^2 - s^2 = n$.

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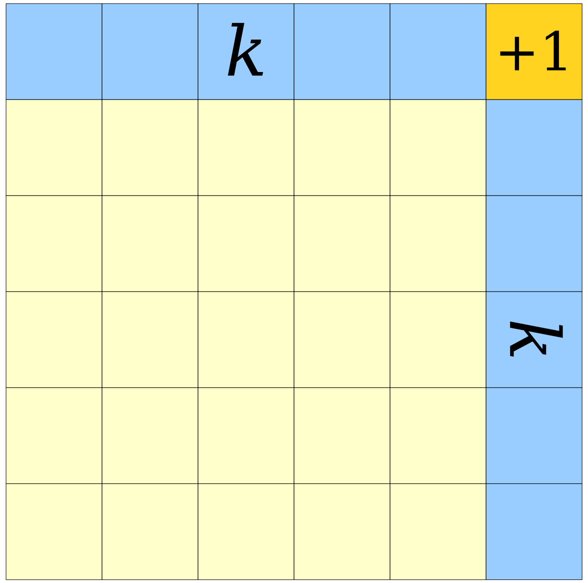

$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For all odd integers n,
there exist integers r and s where $r^2 - s^2 = n$.

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Proof:

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Since n is odd, we know there is an integer k where $n = 2k + 1$.

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$.

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 \quad = \quad (k+1)^2 - k^2$$

Theorem: For all odd integers n, there exist integers r and s where $r^2 – s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 – s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \end{aligned}$$

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Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$
\begin{aligned}
r^2 - s^2 &= (k+1)^2 - k^2 \\
&= k^2 + 2k + 1 - k^2 \\
&= 2k + 1 \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, which is what we needed to show.

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This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n … there … $n = 2k$ … we see that

We ask the reader to make an ***arbitrary choice***.
Rather than specifying what ***n*** is, we're signaling to the reader that they could, in principle, supply any choice ***n*** that they'd like.

$$= \quad 2k + 1$$

$$= \quad n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know th ... re $n = 2k + 1$. Now, let $r = k$ ... ee that

$$r^2 - s^2 = (k+ \ldots$$

$$= k^2 + 2k + 1 - k^2$$

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 \ldots$$
$$= k^2 + 2 \ldots$$
$$= 2k + 1 \ldots$$
$$= n.$$

This means that $r^2 - s^2 = n$, w… to show. ■

We're trying to prove an existential statement. First we announce concrete choices of the objects being sought out.

Theorem: For all odd integers n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick an arbitrary odd integer n. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.

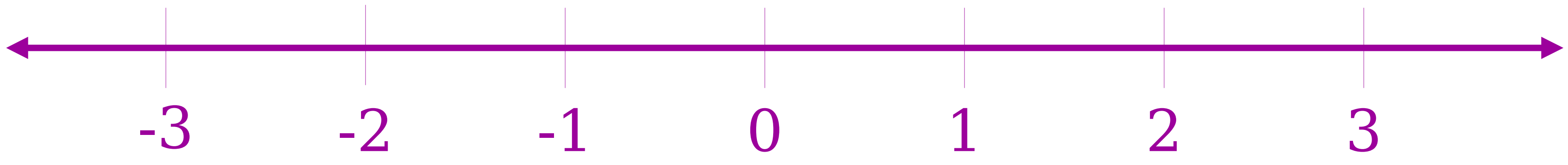
$$\lceil 1 \rceil = 1 \qquad \lceil 1.5 \rceil = 2$$

$$\lceil -1 \rceil = -1 \qquad \lceil -1.5 \rceil = -1$$

- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.

$$\lfloor 1 \rfloor = 1 \qquad \lfloor 1.5 \rfloor = 1$$

$$\lfloor -1 \rfloor = -1 \qquad \lfloor -1.5 \rfloor = -2$$



Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Hmmm…are we stuck? Typically, our third sentence (after the “assume” and “WTS”/”want to show” sentences) is an expansion of a definition word, such as “even.” Here we just have an integer, so there’s no definition to expand.

Let’s take a moment to do some work on scratch paper and see if we can find a way forward?

Let's Try Some Examples!

Scratch paper work.

$$\lceil 0/2 \rceil + \lfloor 0/2 \rfloor = 0 + 0 = 0$$

$$\lceil 1/2 \rceil + \lfloor 1/2 \rfloor = 1 + 0 = 1$$

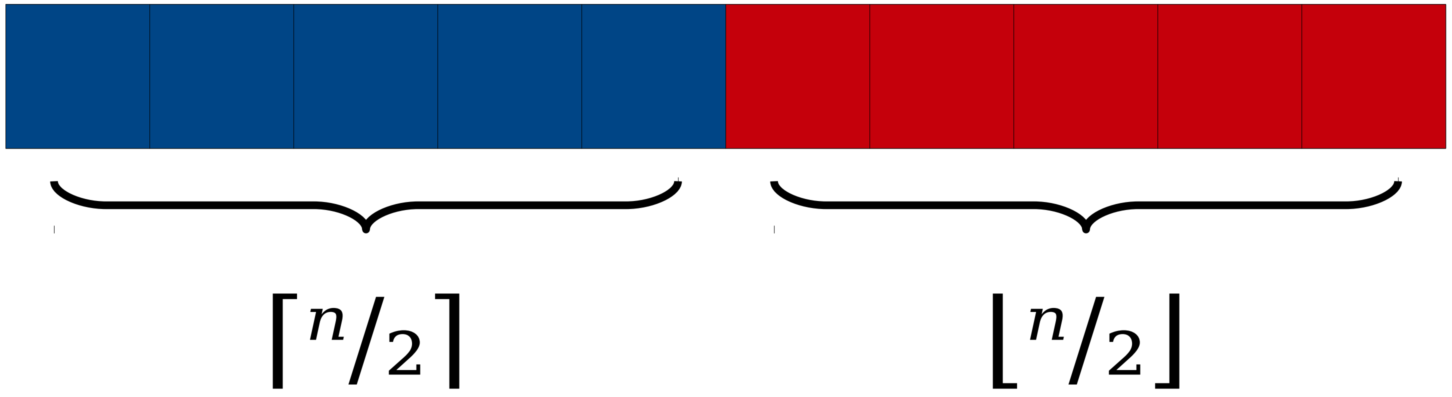
$$\lceil 2/2 \rceil + \lfloor 2/2 \rfloor = 1 + 1 = 2$$

$$\lceil 3/2 \rceil + \lfloor 3/2 \rfloor = 2 + 1 = 3$$

$$\lceil 4/2 \rceil + \lfloor 4/2 \rfloor = 2 + 2 = 4$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let's Draw Some Pictures!



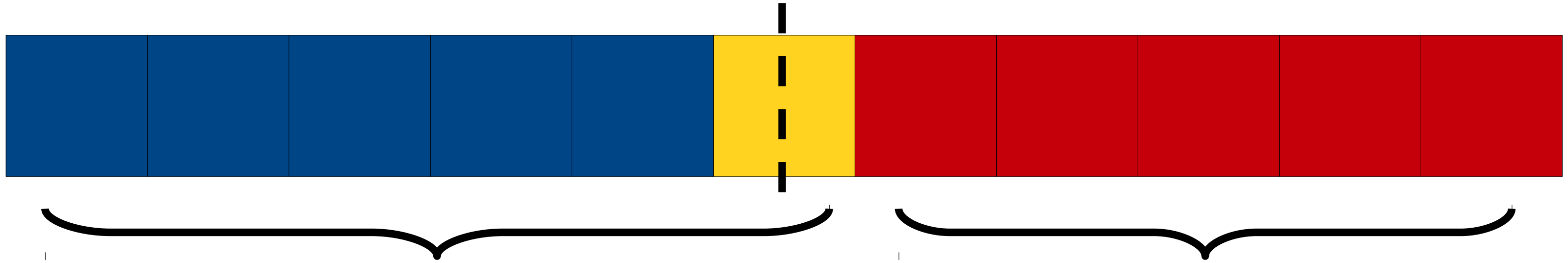
Scratch paper work.

Hm, too bad we weren't asked to this proof for a theorem that says that *n* is even—that looks easy! :-(:-(

$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let's Draw Some Pictures!



$\lceil n/2 \rceil$ $\lfloor n/2 \rfloor$

Scratch paper work.

Hm, too bad we weren't asked to this proof for a theorem that says that *n* is odd—a little harder than even, but we can still find a way. :-(:-(

$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Turns out that we can just smash our two proofs together—the odd one and the even one—and that counts as a proof for *all* integers—yay!!

Thanks, **Proof by Cases**!

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$.

Some algebra then tells us that

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor \\ &= \lceil k \rceil + \lfloor k \rfloor \\ &= 2k \\ &= n. \end{aligned}$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= (k+1) + k \\ &= 2k+1 \\ &= n. \end{aligned}$$

The case labels in effect introduce the new assumptions you wish you had, to make the proof solvable. Then you proceed to show your work from there.

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We want to show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

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$$= \lceil k \rceil + \lfloor k \rfloor$$
$$= 2k$$
$$= n.$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

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In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

Proofs as a Dialog

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Pick an arbitrary odd integer n.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

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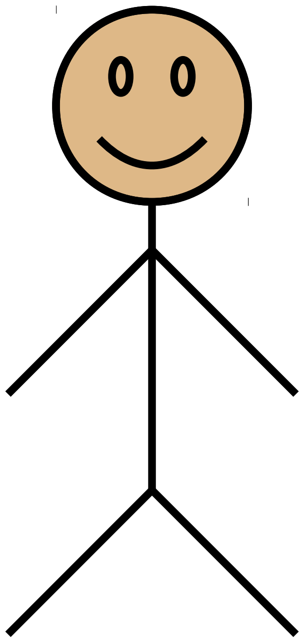

Proof Writer (You)

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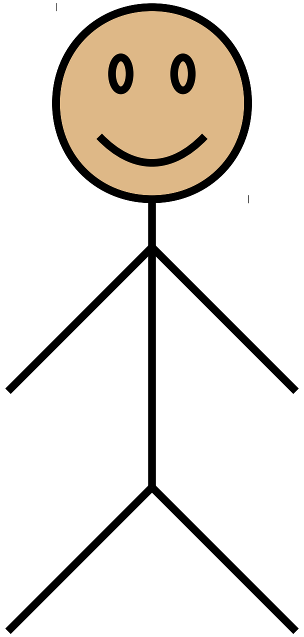

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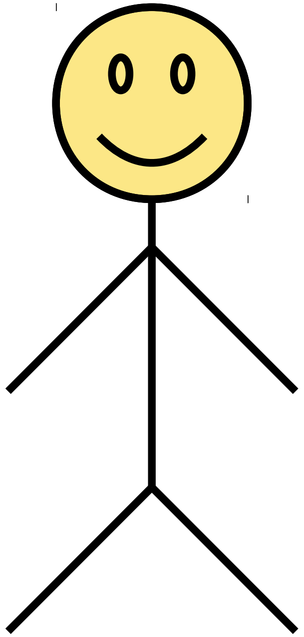

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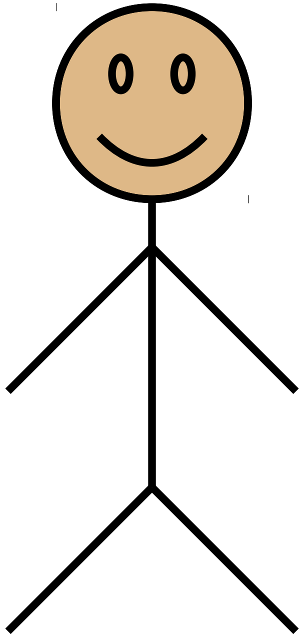

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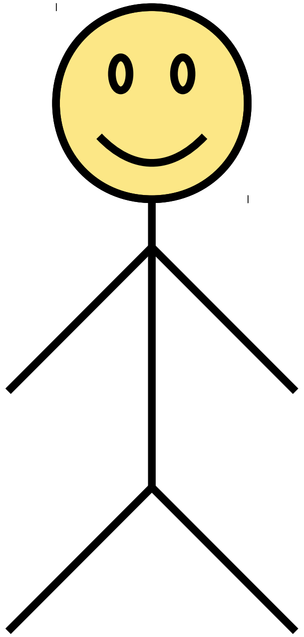

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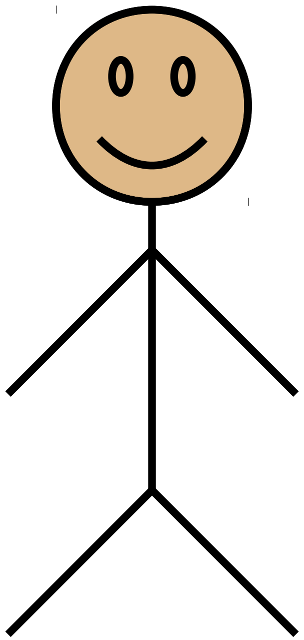
$n = 137$

Reader *Picks*



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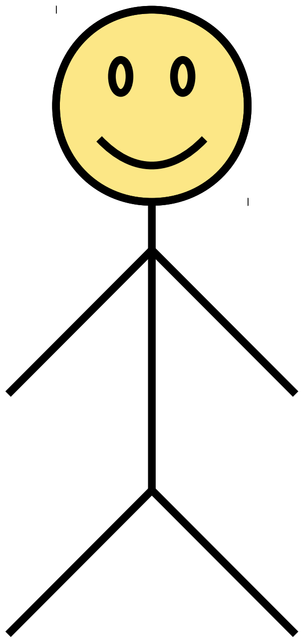

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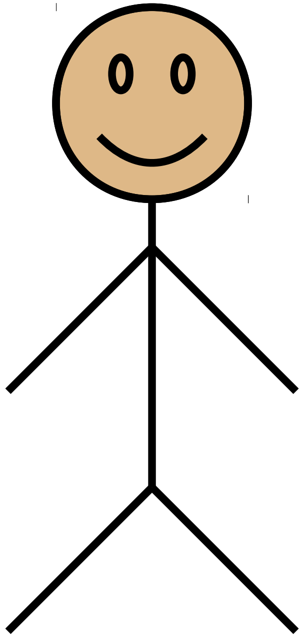

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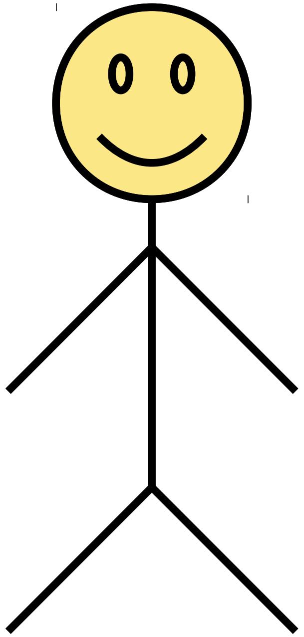

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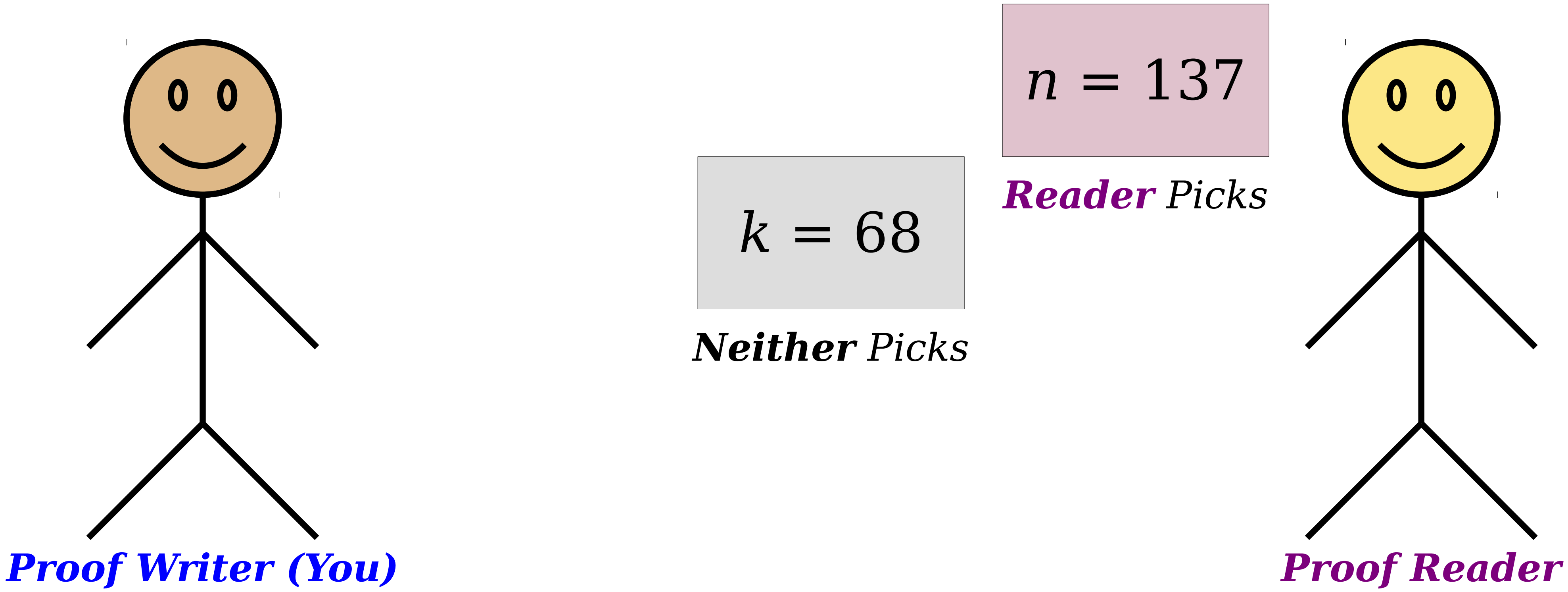

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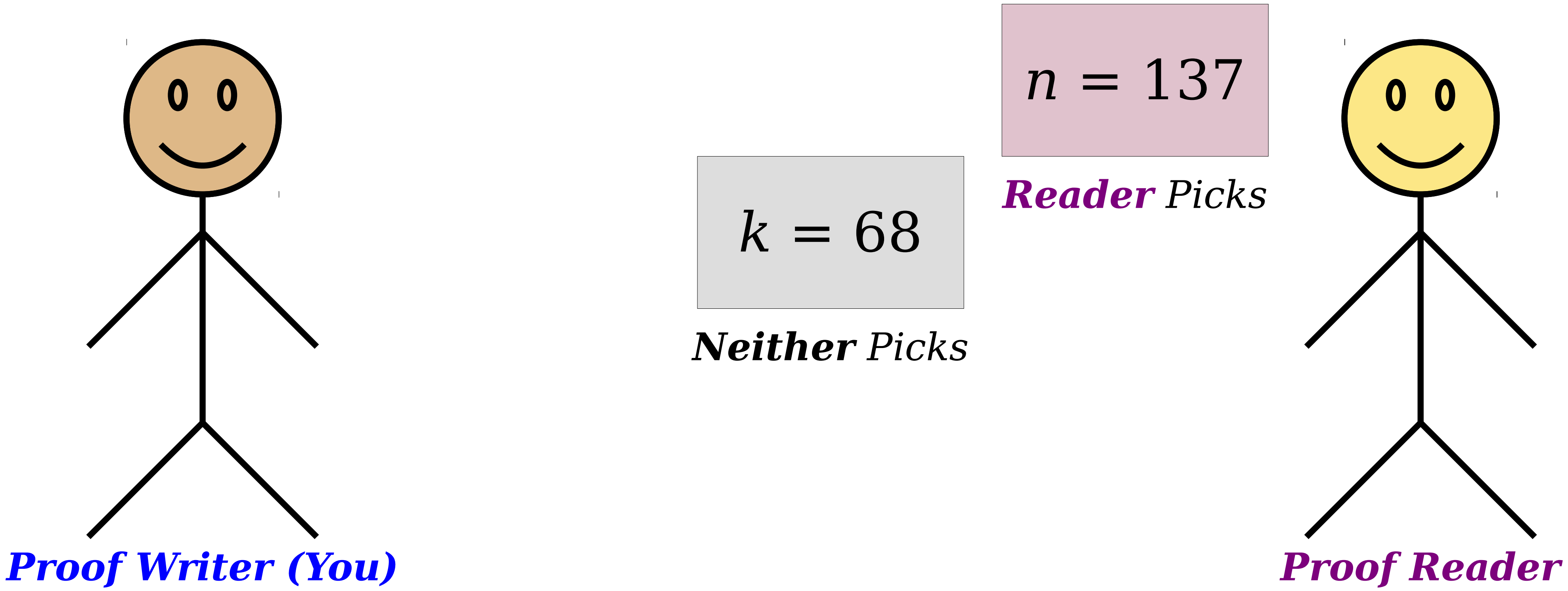
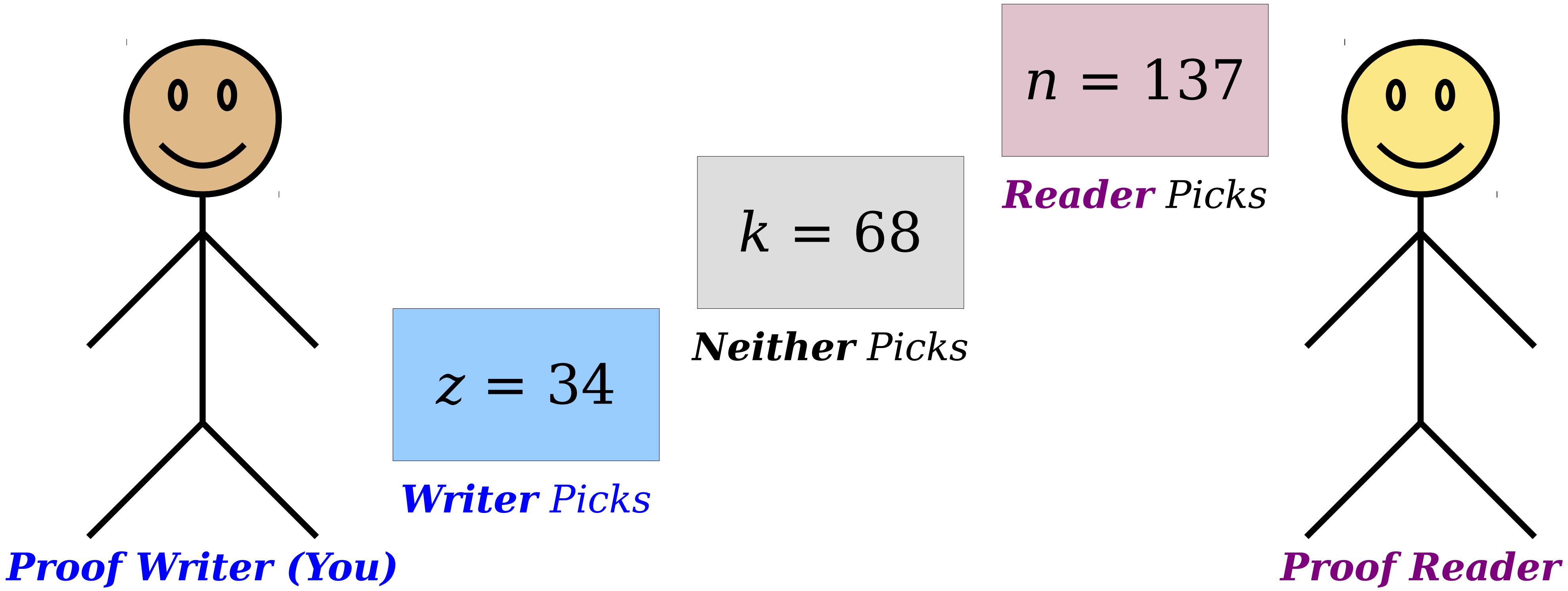


Proofs as a Dialog

Pick an arbitrary odd integer n.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.



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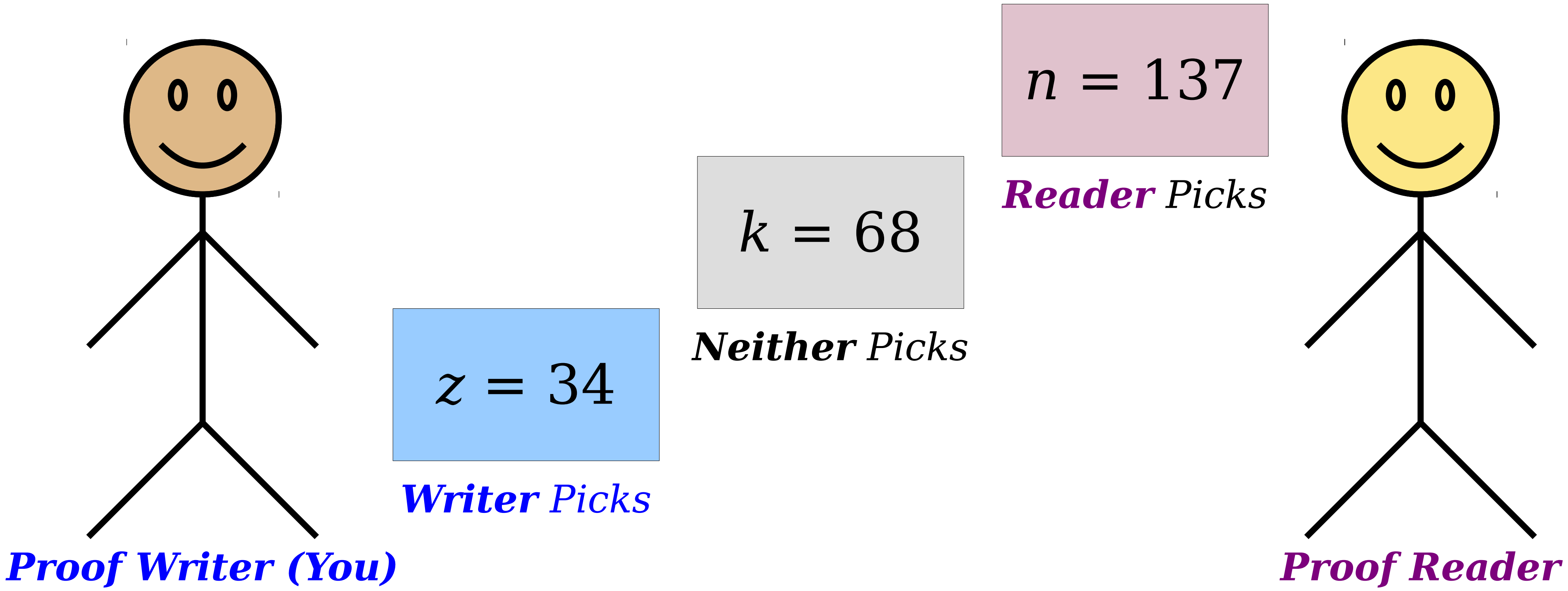

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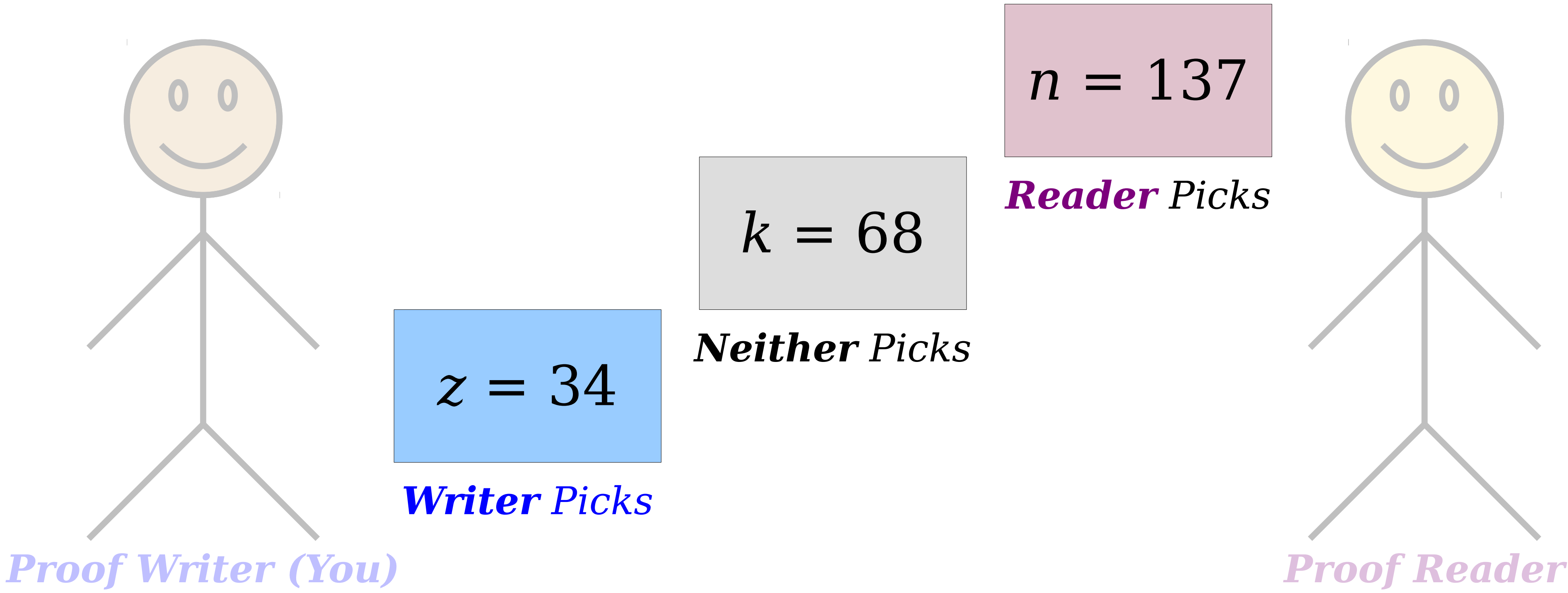
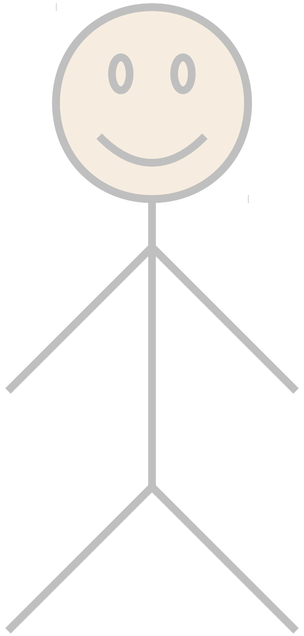


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since … r

Now, let $z = k - 34$.



Proof Writer (You)

$n = 137$

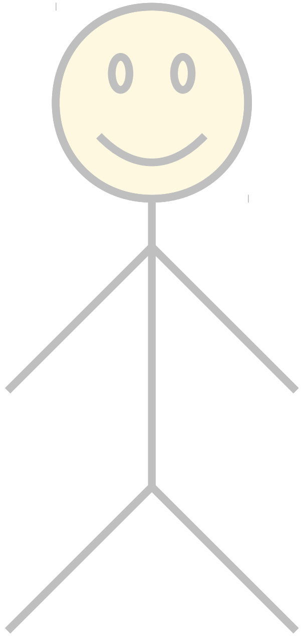
Reader *Picks*

$k = 68$

Neither *Picks*

$z = 34$

Writer *Picks*



Proof Reader

Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick an arbitrary natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Let n be a natural number.
  - Let n be an arbitrary $n \in \mathbb{N}$.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - We choose $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, <u>we know there is</u> some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, <u>there must be some</u> integer k where $n = 2k + 1$.

Proofwriting Rules We Learned Today

- Direct proof: The first two sentences of a proof of a theorem with “If…then…” form are (1) assume the “if” part, (2) announce you “want to show” the “then” part.
- To prove a universal, “pick an arbitrary.”
- To prove an existential, (1) announce a concrete value that works, then (2) justify that it works.
- Use formal definitions of terms.
- Write in complete sentences.
- Clearly introduce variable names using prescribed language. Don’t reuse/overlap variable names.
- Proof by Cases: You may divide a situation into all possible cases, and prove each one separately, to prove the whole. Give clear case labels, which act as assumptions.

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if P, then Q" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.